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Abstract
Quantum mechanics in the non-commutative plane is shown to admit the 'exotic' symmetry of the doubly centrally extended Galilei group. When coupled to a planar magnetic field whose strength is the inverse of the non-commutative parameter, the system becomes singular, and 'Faddeev–Jackiw' reduction yields the 'Chern–Simons' mechanics of Dunne *et al*. The reduced system moves according to the Hall law.

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1. Introduction

Quantum mechanics in the non-commutative plane has been at the center of recent interest [1]. Some formulae in [2] are, in particular, rather similar to those we found in [3], where we started with the two-fold central extension of the planar Galilei group [4–6], labelled by the mass m and the 'exotic' parameter κ. Then we argued that a non-relativistic particle in the plane associated to this 'exotic' Galilei group is endowed with an unconventional structure. Here we point out that quantum mechanics in the non-commutative plane actually admits our 'exotic' Galilean symmetry. The two models are in fact equivalent, the non-commutative parameter θ being related to the 'exotic' one according to

$$\theta = \frac{\kappa}{m^2}. \tag{1.1}$$

Coupling an 'exotic' particle to an electromagnetic field, the two extension parameters combine with the magnetic field B into an effective mass m^* given by (3.4); when this latter vanishes, we found, furthermore, that the consistency of the equations of motion requires that the particle obeys the Hall law [3, 7]. Below, we rederive and generalize these results using the framework of Faddeev and Jackiw [8]. For $m^* = 0$, we get the 'Chern–Simons mechanics' considered some time ago by Dunne *et al* [9].

The reduced theory admits the infinite symmetry of area-preserving diffeomorphisms, found before for the edge currents of the quantum Hall states [10]. Finally, we illustrate the general theory with examples.

2. Exotic symmetry

The fundamental commutation relations for the non-commutative plane [1, 2] are given by

$$
\begin{aligned}
\{x_1, x_2\} &= \theta \\
\{x_i, p_j\} &= \delta_{ij} \\
\{p_1, p_2\} &= 0
\end{aligned}
\tag{2.1}
$$

where θ is the non-commutative parameter. The Poisson bracket on phase space,

$$
\{f, g\} = \frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial g}{\partial \vec{p}} - \frac{\partial g}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} + \theta \left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1} \frac{\partial f}{\partial x_2} \right)
\tag{2.2}
$$

differs hence from the canonical one by an additional term. Hamilton's equations of a free particle, $\dot{\xi}_\alpha = \{\xi_\alpha, h_0\}$ where $h_0 = \vec{p}^{\,2}/(2m)$ and $\xi = (p_1, p_2, x_1, x_2)$, describe therefore the usual free motion. Owing to the extra term in (2.2), some of the conserved quantities contain additional terms. The modified angular momentum and Galilean boosts,

$$
\begin{aligned}
j &= \vec{x} \times \vec{p} + \tfrac{1}{2}\theta \vec{p}^{\,2} + s \\
g_i &= mx_i - p_i t + m\theta\, \varepsilon_{ij}\, p_j
\end{aligned}
\tag{2.3}
$$

(where s is anyonic spin) commute indeed with the free Hamiltonian h_0. The key point is that these quantities satisfy, with the momenta and the energy p_i and h_0, the 'exotic' commutation relations of the doubly extended Galilei group [4, 5], which differ from the standard Galilean commutation relations only in that the boosts close on the exotic parameter (1.1) according to

$$
\{g_1, g_2\} = -m^2\theta.
\tag{2.4}
$$

The Hamiltonian framework presented here is consistent with the acceleration-dependent Lagrangian of Lukierski *et al* [6]. This latter is conveniently presented as

$$
L_0 = \vec{p} \cdot \dot{\vec{x}} - \frac{\vec{p}^{\,2}}{2m} + \frac{\theta}{2}\, \vec{p} \times \dot{\vec{p}}.
\tag{2.5}
$$

Then it is straightforward to show that under a Galilean boost, $\vec{x} \to \vec{x} + \vec{b}t$, $\vec{p} \to \vec{p} + m\vec{b}$, the Lagrangian L_0 merely changes by a total time derivative,

$$
L_0 \to L_0 + m\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{x} \cdot \vec{b} + \tfrac{1}{2}\vec{b}^{\,2}t + \frac{\theta}{2}\vec{b} \times \vec{p} \right)
\tag{2.6}
$$

confirming that the model is indeed non-relativistic. The model based on the Lagrangian (2.5) is indeed equivalent to that constructed in [3–5].

The conserved quantities are readily recovered by Noether's theorem: if an infinitesimal transformation changes the Lagrangian by a total time derivative, $\delta L = \mathrm{d}C/\mathrm{d}t$ (i.e. a symmetry), then $(\partial L/\partial \dot{\xi}_\alpha)\delta\xi_\alpha - h\delta t - C$ is conserved. A rotation leaves L_0 invariant, so that the exotic contribution to the angular momentum in (2.3) comes from the $(\partial L/\partial \dot{\xi}_\alpha)\delta\xi_\alpha$ term alone. For a boost, half of the exotic contribution comes from the latter term, and the other half from the response (2.6) of the Lagrangian.

The doubly centrally extended—or 'exotic'—Galilei group can be conveniently represented by the group of the 6×6 matrices

$$a = \begin{pmatrix} A & \vec{b} & 0 & \vec{c} & \frac{1}{2}\varepsilon\vec{b} \\ 0 & 1 & 0 & e & 0 \\ \vec{b} \cdot A & \frac{1}{2}\vec{b}^2 & 1 & u & v \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

where ε is the matrix (ε_{ij}); $A \in SO(2)$ represents a planar rotation, $\vec{b}$ a Galilean boost, $\vec{c}$ a space translation and e a time translation; u and v parametrize the two-dimensional center. The classical phase space with Poisson structure (2.2) can be identified, as in [3, 5], with a co-adjoint orbit of the doubly extended Galilei group (2.7) defined by the invariants m and κ, cf (1.1).

Due to the presence of the 'exotic' term there is no position representation. Our clue is to observe that the p_i and

$$Q_i = x_i + \tfrac{1}{2}\theta\varepsilon_{ij}p_j \tag{2.8}$$

are canonical coordinates so that they satisfy the *ordinary* relations (2.1) with $\theta = 0$. Hence, canonical quantization yields, in the momentum picture, that the quantum operator $\hat{p}_i$ is multiplication by p_i, and (setting $\hbar = 1$):

$$\hat{x}_j = \hat{Q}_j - \frac{1}{2}\theta\varepsilon_{jk}\,\hat{p}_k = i\frac{\partial}{\partial p_j} - \frac{1}{2}\theta\varepsilon_{jk}p_k. \tag{2.9}$$

Putting $G_i = g_i - (m\theta/2)\varepsilon_{ij}p_j$ we get $\{G_i, G_j\} = 0$, while the other commutation relations remain unchanged: we obtain the ordinary (singly extended) Galilei algebra [5]. Hence, the Hamiltonian $\hat{h}_0 = \vec{p}^2/(2m)$ is standard. Unlike its classical counterpart in (2.3), the quantum angular momentum retains the usual form $\hat{j} = -i\varepsilon_{jk}p_j\partial_{p_k} + s$, whereas the 'exotic' contribution appears only in the boosts, namely

$$\hat{g}_j = m\left[i\frac{\partial}{\partial p_j} + \frac{1}{2}\theta\varepsilon_{jk}\,p_k\right]. \tag{2.10}$$

The factor $1/2$ here w.r.t. the classical expression (2.3) is explained by $\hat{g}_i = m\hat{x}_i + m\theta\varepsilon_{ij}p_j = m\hat{Q}_i + \frac{1}{2}m\theta\varepsilon_{ij}p_j$. Completed with the mass m and the 'exotic' parameter $\kappa = m^2\theta$ these operators span the 'exotic' Galilei algebra [4, 5].

The associated irreducible unitary representation $U_{m,\theta}$ of the matrix group (2.7), on the space of wavefunctions $\psi(\vec{p})$, is deduced accordingly,

$$U_{m,\theta}(a)\psi(\vec{p}) = \exp\left(i\left[\frac{\vec{p}^2e}{2m} - \vec{p}\cdot\vec{c} + s\varphi + m\,u\right]\right.$$
$$\left. + im\theta\left[\frac{1}{2}\vec{b}\times\vec{p} + mv\right]\right)\psi(A^{-1}(\vec{p} - m\vec{b})) \tag{2.11}$$

where φ is the angle of the rotation A (see also [5]).

3. Coupling to a gauge field

As found in [3], using Souriau's symplectic framework [11], minimal coupling to an electromagnetic field (E, B) unveils new and surprising features. Here we explain this using

the 'Faddeev–Jackiw' formalism [8]. Let us hence generalize the free expression (2.5) by considering the action

$$\int (\vec{p} - \vec{A}) \cdot d\vec{x} - h \, dt + \frac{\theta}{2} \vec{p} \times d\vec{p},$$ (3.1)

where $(V, \vec{A})$ is an electro-magnetic potential, the Hamiltonian being given by

$$h = \frac{\vec{p}^{\,2}}{2m} + V.$$ (3.2)

The associated Euler–Lagrange equations read

$$m^* \dot{x}_i = p_i - m\theta \, \varepsilon_{ij} E_j$$
$$\dot{p}_i = E_i + B \, \varepsilon_{ij} \dot{x}_j$$ (3.3)

where we have introduced the *effective mass*

$$m^* = m(1 - \theta B).$$ (3.4)

The velocity and momentum are different if $\theta \neq 0$. The equations of motions (3.3) can also be written as

$$\omega_{\alpha\beta} \dot{\xi}_\beta = \frac{\partial h}{\partial \xi_\alpha} \qquad \text{where} \quad (\omega_{\alpha\beta}) = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}.$$ (3.5)

Note that the electric and magnetic fields are otherwise arbitrary solutions of the homogeneous Maxwell equation $\partial_t B + \varepsilon_{ij} \partial_i E_j = 0$, which guarantees that the two-form $\omega = \frac{1}{2}\omega_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta$ is closed, $d\omega = 0$. Our matrix (3.5) is in fact (m^*/m) times that posited in [2].

When $m^* \neq 0$, the determinant $\det(\omega_{\alpha\beta}) = (1 - \theta B)^2 = (m^*/m)^2$ is nonzero; the matrix $(\omega_{\alpha\beta})$ in (3.5) is indeed symplectic, and can therefore be inverted. Then the equations of motion (3.5) (or (3.3)) take the form $\dot{\xi}_\alpha = \{\xi_\alpha, h\}$, with the standard Hamiltonian (3.2), but with the new Poisson bracket $\{f, g\} = (\omega^{-1})_{\alpha\beta} \partial_\alpha f \partial_\beta g$ which reads explicitly

$$\{f, g\} = \frac{m}{m^*} \left[\frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial g}{\partial \vec{p}} - \frac{\partial g}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} + \theta \left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1} \frac{\partial f}{\partial x_2} \right) + B \left(\frac{\partial f}{\partial p_1} \frac{\partial g}{\partial p_2} - \frac{\partial g}{\partial p_1} \frac{\partial f}{\partial p_2} \right) \right].$$ (3.6)

Note that the fundamental commutation relations (2.1) are now modified [3] as

$$\{x_1, x_2\} = \frac{m}{m^*} \theta$$
$$\{x_i, p_j\} = \frac{m}{m^*} \delta_{ij}$$ (3.7)
$$\{p_1, p_2\} = \frac{m}{m^*} B.$$

Thus, both the coordinates and the momenta span independent Heisenberg algebras.

Further insight can be gained when the magnetic field B is a (positive) nonzero constant, which turns out the most interesting case, and will be henceforth assumed. The vector

potential can then be chosen as $A_i = \frac{1}{2}B\varepsilon_{ij}x_j$, the electric field $E_i = -\partial_i V$ being still arbitrary. Introducing the new coordinates somewhat similar to those in (2.8),

$$Q_i = x_i + \frac{1}{B}\left[1 - \sqrt{\frac{m^*}{m}}\right]\varepsilon_{ij}p_j$$

$$P_i = \sqrt{\frac{m^*}{m}}\,p_i - \frac{1}{2}B\varepsilon_{ij}\,Q_j \tag{3.8}$$

will allow us to generalize our results in [3] from a constant to any electric field.

Firstly, the 'Cartan' one-form [11] in the action (3.1) reads simply $P_i\,\mathrm{d}Q_i - h\,\mathrm{d}t$, so that the symplectic form on phase space retains the canonical guise $\omega = \mathrm{d}P_i \wedge \mathrm{d}Q_i$. The price to pay is that the Hamiltonian becomes rather complicated,

$$h = \frac{1}{2m^*}\left(\vec{P} + \frac{1}{2}B\varepsilon\vec{Q}\right)^2 + V\left(\alpha\vec{Q} + \beta\varepsilon\vec{P}\right) \tag{3.9}$$

with $\alpha = \frac{1}{2}\left(1 + \sqrt{m/m^*}\right)$ and $\beta = B^{-1}\left(1 - \sqrt{m/m^*}\right)$.

The equations of motion (3.3) are conveniently presented in terms of the new variables $\vec{Q}$ and the old momenta $\vec{p}$ as

$$\dot{Q}_i = \varepsilon_{ij}\frac{E_j}{B} + \sqrt{\frac{m}{m^*}}\left(\frac{p_i}{m} - \varepsilon_{ij}\frac{E_j}{B}\right)$$

$$\dot{p}_i = \varepsilon_{ij}B\frac{m}{m^*}\left(\frac{p_j}{m} - \varepsilon_{jk}\frac{E_k}{B}\right). \tag{3.10}$$

Note that all these expressions diverge when m^* tends to zero.

When the magnetic field takes the particular value

$$B = B_{\mathrm{c}} = \frac{1}{\theta} \tag{3.11}$$

the effective mass (3.4) vanishes, $m^* = 0$, so that $\det(\omega_{\alpha\beta}) = 0$ and the system becomes singular. Then the time derivatives $\dot{\xi}_\alpha$ can no longer be expressed from the variational equations (3.5) and we have resort to 'Faddeev–Jackiw' reduction [8]. In accordance with the Darboux theorem (see e.g. [11]), the Cartan one-form in (3.1) can be written, up to an exact term, as $\vartheta - h\,\mathrm{d}t$ with $\vartheta = \left(p_i - \frac{1}{2}B_{\mathrm{c}}\varepsilon_{ij}x_j\right)\mathrm{d}x_i + \frac{1}{2}\theta\varepsilon_{ij}p_i\,\mathrm{d}p_j = P_i\,\mathrm{d}Q_i$, where the new coordinates read, consistently with (3.8),

$$Q_i = x_i + \frac{1}{B_{\mathrm{c}}}\varepsilon_{ij}p_j \tag{3.12}$$

while the $P_i = -\frac{1}{2}B_{\mathrm{c}}\varepsilon_{ij}Q_j$ are in fact the rotated coordinates Q_i. Eliminating the original coordinates $\vec{x}$ using (3.12), we see that the Cartan one-form reads $P_i\,\mathrm{d}Q_i - H(\vec{Q},\vec{p})\,\mathrm{d}t$, where $H(\vec{Q},\vec{p}) = \vec{p}^2/(2m) + V(\vec{Q},\vec{p})$. As the p_i appear here with no derivatives, they can be eliminated using their equation of motion $\partial H(\vec{Q},\vec{p})/\partial\vec{p} = 0$, namely

$$\frac{p_i}{m} - \frac{\varepsilon_{ij}E_j}{B_{\mathrm{c}}} = 0 \tag{3.13}$$

cf (3.10). Inserting (3.13) into (3.12) and taking partial derivatives we find

$$\frac{\partial Q_j}{\partial x_i} = \delta_{ji} - \frac{m}{B_{\mathrm{c}}^2}\frac{\partial E_j}{\partial x_i} \qquad \frac{\partial H}{\partial x_i} = m\frac{E_j}{B_{\mathrm{c}}}\frac{\partial E_j}{\partial x_i} - E_i.$$

Hence $\partial H/\partial\vec{Q} = (\partial H/\partial\vec{x})\cdot(\partial\vec{x}/\partial\vec{Q}) = -\vec{E}$. Consequently, the reduced Hamiltonian is (modulo a constant) just the original potential, viewed as a function of the 'twisted' coordinates $\vec{Q}$, namely

$$H = V(\vec{Q}). \tag{3.14}$$

This rule is referred to as the 'Peierls substitution' [3, 9]. Since $\partial^2 H/\partial p_i \partial p_j = \delta_{ij}/m$ is already non-singular, the reduction stops and we end up with the reduced Lagrangian

$$L_{\text{red}} = \frac{1}{2\theta} \vec{Q} \times \dot{\vec{Q}} - V(\vec{Q}) \tag{3.15}$$

supplemented with the Hall constraint (3.13). The four-dimensional phase space is hence reduced to two dimensions, with Q_1 and Q_2 in (3.12) as canonical coordinates and reduced symplectic two-form $\omega_{\text{red}} = \frac{1}{2} B_c \varepsilon_{ij} \, dQ_i \wedge dQ_j$ so that the reduced Poisson bracket is

$$\{F, G\}_{\text{red}} = -\frac{1}{B_c} \left(\frac{\partial F}{\partial Q_1} \frac{\partial G}{\partial Q_2} - \frac{\partial G}{\partial Q_1} \frac{\partial F}{\partial Q_2} \right). \tag{3.16}$$

The twisted coordinates are therefore again non-commuting,

$$\{Q_1, Q_2\}_{\text{red}} = -\theta = -\frac{1}{B_c} \tag{3.17}$$

cf (2.4). The equations of motion associated with (3.15) and also consistent with the Hamilton equations $\dot{Q}_i = \{Q_i, H\}_{\text{red}}$ are given by

$$\dot{Q}_i = \varepsilon_{ij} \frac{E_j}{B_c} \tag{3.18}$$

in accordance with the Hall law (compare (3.10) with the divergent terms removed).

Putting $B_c = 1/\theta$, the Lagrangian (3.15) becomes formally identical to the one Dunne *et al* [9] derived letting the *real* mass go to zero. Note, however, that while $\vec{Q}$ denotes real position in [9], our $\vec{Q}$ here is the 'twisted' expression (3.12), with the magnetic field frozen at the critical value $B_c = 1/\theta$, determined by the 'exotic' structure.

4. Quantization

Let us conclude our general theory by the quantization of the coupled system. Again, owing to the exotic term, the position representation does not exist. We can use, instead, the twisted coordinates $\vec{Q}$ in (3.8); and consider wavefunctions as simply depending on $\vec{Q}$. Quantizing the Hamiltonian (3.9) is, however, a rather tough task: apart from the 'gentle' quadratic kinetic term, one also has to quantize the otherwise arbitrary function $V(\alpha \vec{Q} + \beta \varepsilon \vec{p})$ of the conjugate variables $\vec{p}$ and $\vec{Q}$. This goes beyond our scope here; we focus, therefore, our attention to the kinetic term.

Introducing the complex coordinates

$$z = \frac{\sqrt{B}}{2}(Q_1 + iQ_2) + \frac{1}{\sqrt{B}}(-iP_1 + P_2)$$
$$w = \frac{\sqrt{B}}{2}(Q_1 - iQ_2) + \frac{1}{\sqrt{B}}(-iP_1 - P_2) \tag{4.1}$$

the two-form $dP_i \wedge dQ_i$ on four-dimensional phase space becomes the canonical Kähler two-form of $\mathbf{C}^2$, namely $\omega = (2i)^{-1} (d\bar{z} \wedge dz + d\bar{w} \wedge dw)$. Then geometric quantization [11, 12] yields, with the choice of the antiholomorphic polarization, the 'unreduced' quantum Hilbert space, consisting of the 'Bargmann–Fock' wavefunctions

$$\psi(z, \bar{z}, w, \bar{w}) = f(z, w) e^{-\frac{1}{4}(z\bar{z} + w\bar{w})} \tag{4.2}$$

where f is holomorphic in both of its variables. The fundamental quantum operators,

$$\hat{z}f = zf \quad \hat{\bar{z}}f = 2\partial_z f$$
$$\hat{w}f = wf \quad \hat{\bar{w}}f = 2\partial_w f \tag{4.3}$$

satisfy the commutation relations $\left[\hat{\bar{z}}, \hat{z}\right] = \left[\hat{\bar{w}}, \hat{w}\right] = 2$, and $\left[\hat{z}, \hat{w}\right] = \left[\hat{\bar{z}}, \hat{w}\right] = 0$. We recognize here the familiar creation and annihilation operators, namely $a_z^* = z, a_w^* = w$ and $a_z = \partial_z, a_w = \partial_w$.

Using (3.8), the (complex) momentum $p = p_1 + \mathrm{i}p_2$ and the kinetic part h_0 of the Hamiltonian (3.2) become, respectively,

$$p = -\mathrm{i}\sqrt{\frac{mB}{m^*}}\,\bar{w} \qquad \text{and} \qquad h_0 = \frac{B}{2m^*}\,w\bar{w}. \tag{4.4}$$

For $m^* \neq 0$ the wavefunction satisfies the Schrödinger equation $\mathrm{i}\partial_t f = \hat{h}f$, with $\hat{h} = \hat{h}_0 + \hat{V}$. The quadratic kinetic term here is

$$\hat{h}_0 = \frac{B}{4m^*}\left(\hat{w}\,\hat{\bar{w}} + \hat{\bar{w}}\,\hat{w}\right) = \frac{B}{2m^*}\left(\hat{w}\,\hat{\bar{w}} + 1\right). \tag{4.5}$$

The case when the effective mass tends to zero is conveniently studied in this framework. On the one hand, in the limit $m^* \to 0$, one has

$$z \to \sqrt{B}\,Q \qquad w \to 0 \tag{4.6}$$

where $Q = Q_1 + \mathrm{i}Q_2$, cf (3.8); the four-dimensional phase space reduces to the complex plane. On the other hand, from (4.4) and (4.3) we deduce that

$$\mathrm{i}\sqrt{\frac{m^*}{mB}}\,\hat{p} = \hat{\bar{w}} = 2\partial_w. \tag{4.7}$$

The limit $m^* \to 0$ is hence enforced, at the quantum level, by requiring that the wavefunctions be independent of the coordinate w, that is

$$\partial_w f = 0 \tag{4.8}$$

yielding the reduced wavefunctions of the form

$$\Psi(z, \bar{z}) = f(z)\mathrm{e}^{-\frac{1}{4}z\bar{z}} \tag{4.9}$$

where f is a holomorphic function of the reduced phase space parametrized by z. When viewed in the 'big' Hilbert space (see (4.2)), these wavefunctions belong, by (4.5), to the lowest Landau level [3, 7, 13].

Using the fundamental operators $\hat{z}$ and $\hat{\bar{z}}$ given in (4.3), we easily see that the (complex) 'physical' position $x = x_1 + \mathrm{i}x_2$ and its quantum counterpart $\hat{x}$, namely

$$x = \frac{1}{\sqrt{B_c}}\left(z + \sqrt{\frac{m}{m^*}}\,\bar{w}\right) \qquad \hat{x} = \frac{1}{\sqrt{B_c}}\left(z + \sqrt{\frac{m}{m^*}}\,2\partial_w\right) \tag{4.10}$$

manifestly diverge when $m^* \to 0$. *Positing from the outset* of the conditions (4.8) the divergence is suppressed, however, leaving us with the reduced position operators

$$\hat{x}f = \hat{Q}f = \frac{1}{\sqrt{B_c}}\,zf \qquad \hat{\bar{x}}f = \hat{\bar{Q}}f = \frac{2}{\sqrt{B_c}}\,\partial_z f \tag{4.11}$$

whose commutator is $[\hat{Q}, \hat{\bar{Q}}] = 2/B_c$, cf (3.17). In conclusion, we recover the 'Laughlin' description [7] of the ground states of the FQHE. (In [3], these results have been obtained by quantizing the reduced model.) Quantization of the reduced Hamiltonian (which is, indeed, the potential $V(z, \bar{z})$), can be achieved using, for instance, anti-normal ordering [7, 9, 13].

5. Examples

In both the examples studied below, we will consider a particle with unit mass $m = 1$ (and unit charge, as before).

The simplest non-trivial example is provided by a constant electric field [3]. For non-vanishing effective mass, $m^* \neq 0$, the equations of motion (3.10) readily imply that the 'position'

$$R = R_1 + iR_2 \qquad \text{where} \quad R_i = Q_i - \frac{1}{B}\varepsilon_{ij}E_j t \tag{5.1}$$

(as well as the momentum $\vec{p}$) rotates in the plane with frequency B/m^*, namely $R(t) = e^{-i(B/m^*)t}R_0$. In the twisted coordinates $Q_i = R_i + \varepsilon_{ij}(E_j/B)t$, the motion is therefore the usual cyclotronic motion (with modified frequency), while the guiding center drifts with the Hall velocity $\varepsilon_{ij}E_j/B$.

When $m^* = 0$, the reduced equation (3.18) requires simply $\dot{R}_i = 0$: the rotation is eliminated, and we are left, cf (3.13), with the uniform drift of the guiding center alone,

$$x_i(t) = \varepsilon_{ij}\frac{E_j}{B}t + x_i(0). \tag{5.2}$$

The canonical transformations of the reduced Poisson bracket (3.16) coincide with the symplectic transformations of the plane. These latter are in fact generated by the observables, that is, the (smooth) functions $F(R)$ of the variable R in (5.1). But, owing to the particular time dependence of R in (5.1), $\{F, H\} = \partial_t F$ for *any* function $F(R)$, which generates, therefore, a symmetry. In the plane, symplectic and area-preserving transformations coincide, yielding the w_∞ symmetry [10].

Examples of observables F linear in Q include the reduced energy $H = -\vec{E} \cdot \vec{Q}$ and the reduced momenta $\Pi_i = B\varepsilon_{ij}Q_j + E_j t$. These latters have the Poisson bracket of 'magnetic translations', $\{\Pi_1, \Pi_2\}_{\text{red}} = B_c$ (see (3.7)). The quadratic observables generate, in turn, the well-known sp (1)-symmetry of the one-dimensional harmonic oscillator.

It is worth pointing out that the reduced Hilbert space will be acted upon by W_∞, the quantum version of the classical symmetry algebra w_∞.

As another illustration, let us describe an 'exotic' particle moving in a constant magnetic field B and a harmonic potential $V(\vec{x}) = \frac{1}{2}\omega^2\vec{x}^2$, cf [2, 9]. Let us first assume that the effective mass does not vanish (i.e. $m^* \neq 0$). The equations of motion (3.3), namely

$$m^*\ddot{x}_i = B^*\epsilon_{ij}\dot{x}_j - \omega^2 x_i \qquad \text{where} \quad B^* = B + \theta\omega^2 \tag{5.3}$$

describe an ordinary, non-exotic, particle with (effective) mass m^*, moving in a combined 'effective magnetic field' B^* and harmonic field $\vec{E} = -\omega^2\vec{x}$. Our particle evolves according to

$$x(t) = e^{-i(B^*/2m^*)t}\left[C\cos\omega^*t + D\sin\omega^*t\right] \qquad \text{with} \quad \omega^* = \sqrt{\left(\frac{B^*}{2m^*}\right)^2 + \frac{\omega^2}{m^*}} \tag{5.4}$$

where C and D are complex constants; see also [9]. The elliptic trajectories described in the square bracket are hence combined with a circular motion represented by the exponential factor.

The system is plainly symmetric with respect to planar rotations; the conserved angular momentum (consistent with that in [2]) reads

$$j = \vec{x} \times \vec{p} + \frac{\theta}{2}\vec{p}^2 + \frac{B}{2}\vec{x}^2 + s. \tag{5.5}$$

Note here the term coming from the 'exotic' structure, and also the 'spin from isospin' contribution due to the symmetric magnetic field. As to quantization, it is enough to replace B and m by B^* and m^* in the formulae of Dunne *et al* [9].

When the magnetic field takes the critical value $B_c = 1/\theta$, the effective mass vanishes and the motion obeys the reduced equation. The 'twisted' coordinates in (3.12) are now proportional to the original 'physical' position, $\vec{Q} = (1 + \theta^2\omega^2)\vec{x}$. Hence, the motion is governed by the same equations, namely

$$\dot{Q}_i = -\omega_c^* \,\varepsilon_{ij} Q_j \qquad \text{where} \quad \omega_c^* = \frac{\theta\omega^2}{1 + \theta^2\omega^2}. \tag{5.6}$$

Putting $Q = Q_1 + iQ_2$, we find

$$Q(t) = e^{-i\omega_c^* t} Q_0 \tag{5.7}$$

with Q_0 a complex constant. All particles move collectively, namely along circles perpendicular to the electric field, with uniform angular velocity ω_c^*. Intuitively, for $m^* = 0$, the general elliptic trajectories (5.4) are forbidden, leaving us with the simple circular motions only. The reduced symplectic form and Hamiltonian are, respectively,

$$\Omega = \frac{1}{\theta}\,\mathrm{d}Q_1 \wedge \mathrm{d}Q_2 \qquad \text{and} \qquad H = \frac{\omega^2}{2\left(1 + \theta^2\omega^2\right)}\,\vec{Q}^{\,2}. \tag{5.8}$$

Since the Hall constraint (3.13) is consistent with rotational symmetry, the reduced system will have a conserved angular momentum (which turns out to be proportional to the reduced Hamiltonian). Hence, we get, once again, a one-dimensional harmonic oscillator with, this time, the usual quadratic Hamiltonian. Its spectrum is, therefore [9],

$$E_n = \frac{\theta\omega^2}{1 + \theta^2\omega^2}\left(n + \frac{1}{2}\right) \qquad n = 0, 1, \ldots \tag{5.9}$$

At last, the w_∞ symmetry of the reduced model discussed above is now generated by the functions $F(Q_0)$ (see (5.7)).

6. Discussion

Our approach allows us to rederive some previous formulae starting with first principles. If one accepts that any theory consistent with the (extended) Galilean symmetry is physical, there is no reason to discard our 'exotic' Galilean theory. The rather obvious equivalence of 'non-commutative' and 'exotic' approaches found here is important, as it allows for a 'technology transfer'.

The interplay between the 'exotic' and the magnetic terms in (3.6) which leads, for $B = B_c$, to the singular behaviour studied above happens precisely when $\vec{p} \to -\vec{x}/\theta$, $\vec{x} \to \theta\vec{p}$ is a canonical transformation that merely interchanges the magnetic and the 'exotic' terms in (3.5).

It is also rather intriguing to observe that the Hall motions discussed above are, strictly speaking, not the only possible motions of the system. Let us indeed assume that $m^* = 0$, and consider an 'exotic' particle whose initial velocity is *inconsistent* with the Hall law, that is, such that $\pi_i = (p_i)_0 - (1/B)\varepsilon_{ij} E_j(x_0, t_0) \neq 0$. No Hall motion can start with such initial conditions; the consistency of the equations of motion (i.e. $(\dot{\vec{p}}, \dot{\vec{x}}, \dot{t})$ lies in the one-dimensional kernel of the singular two-form $\omega - \mathrm{d}h \wedge \mathrm{d}t$) can, however, be maintained requiring $t = t_0 = \text{const}$. Thus, from each such 'forbidden' point starts a strange, 'instantaneous motion'. For a constant electric field, for example, these 'instantaneous motions' are circles with radius

$|\vec{\pi}|/B_{\mathrm{c}}$, centered at $(x_i)_0 + (1/B_{\mathrm{c}})\varepsilon_{ij}\pi_j$. The physical interpretation of these 'motions' is still unclear to us; our conjecture is that they could be related to the edge motions in the FQHE [7].

While this paper was being completed, there appeared an article [14] discussing rather similar issues. Let us briefly indicate the relation to our paper. In [14] the authors start, following [9], with an ordinary charged particle in a planar magnetic field, and then set the mass to zero. Their model becomes non-commutative only after taking the limit $m \to 0$, and their magnetic field is arbitrary. Here, we start with non-commuting the coordinates, and fine-tune the magnetic field to yield vanishing effective mass, $m^* = 0$. The observation in [14] saying that some functions have vanishing Poisson bracket with the dynamical variables is consistent with our two-form (3.5) becoming singular. Then Guralnik *et al* present a non-commutative magnetohydrodynamical model, analogous to ours in the second paper in [3]. Their constraint $\vec{\pi} = 0$ corresponds to our lowest Landau level condition (4.8).

The recent paper [15] contains also some similar results; they use another system of canonical coordinates.

Acknowledgments

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